4. Integration

- Integration is the inverse process of differentiation. If $\frac{d}{dx} f(x) = g(x)$, then we can write $\int g(x)\,dx = f(x) + C$. This is called the general or the indefinite integral and C is called the constant of integration.
- Some standard indefinite integrals are given as follows:
    - $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
    - $\int dx = x + C$
    - $\int \sin x\,dx = -\cos x + C$
    - $\int \cos x\,dx = \sin x + C$
    - $\int \sec^2 x\,dx = \tan x + C$
    - $\int \operatorname{cosec}^2 x\,dx = -\cot x + C$
    - $\int \sec x \tan x\,dx = \sec x + C$
    - $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$
    - $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$ or $-\cos^{-1} x + C$
    - $\int \frac{dx}{\sqrt{1-x^2}} = \tan^{-1} x + C$ or $-\cot^{-1} x + C$
    - $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$ or $-\operatorname{cosec}^{-1} x + C$
    - $\int e^x dx = e^x + C$
    - $\int a^x dx = \frac{a^x}{\log a} + C$
    - $\int \frac{1}{x} dx = \log |x| + C$
    - $\int e^{ax} dx = \frac{e^{ax}}{a} + C$
- Properties of indefinite integrals:
    - $\frac{d}{dx} \int f(x)\,dx = f(x)$ and $\int f'(x)\,dx = f(x) + C$
    - If the derivative of two indefinite integrals is the same, then they belong to same family of curves and hence they are equivalent.
    - $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
    - $\int k f(x)\,dx = k \int f(x)\,dx$, where k is any constant

- There are three important methods of integration, namely, integration by substitution, integration using partial fractions, and integration by parts.
- **Integration by substitution:** A change in the variable of integration often reduces an integral to one of the fundamental integrals, which can be easily found out. The method in which we change the variable to some other variable is called the method of substitution.
- Using substitution method of integration, we obtain the following standard

  integrals:

    - $\int \tan x\,dx = -\log |\cos x| + C$ or $\log|\sec x| + C$
    - $\int \cot x\,dx = \log |\sin x| + C$

- $\int \sec x\,dx = \log|(\sec x + \tan x)| + C$
- $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x)| + C$

Integration by parts: For given functions $f(x)$ and $g(x)$, $\int f(x)\cdot g(x)\,dx = f(x)\int g(x)\,dx - \int[f'(x)\cdot\int g(x)\,dx]\,dx$

In other words, the integral of the product of two functions is equal to first function × integral of the second function – integral of {differential of the first function × integral of the second function}.
Here, the functions *f* and *g* have to be taken in proper order with respect to the ILATE rule, where I, L, A, T, and E respectively represent inverse, logarithm, arithmetic, trigonometric, and exponential function.

Example: Evaluate $\int x^2 \sin^{-1} x\,dx$

Solution:
Integrating by parts, taking $\sin^{-1} x$ as the first function, we get
sin-1xx33-∫11-x2x33dx
=x33sin-1x-13∫x31-x2dx

Let I ==x33sin-1x-13∫x31-x2dxwhere I1=x33sin-1x , I2=13∫x31-x2dxTherefore, I = I1+ I2
let 1-x2=t1-t=x2-dt=2xdx

Putting these values in the equation we get,
I2=-13∫-(1-t)tdt2=13×12∫1t-ttdt=162t12 -23t32

I = x33sin-1x + 131-x2-19(1-x2)32 + C

- Integration by partial fractions: The following table shows how a function of the form $\frac{P(x)}{Q(x)}$, where $Q(x) \neq 0$ and degree of $Q(x)$ is greater than the degree of $P(x)$, is broken by the concept of partial fractions. After doing this, we find the integration of the given function by integrating the right hand side (i.e., partial fractional form).

Function	Form of partial fraction
$\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)\left(x^2+bx+c\right)}$, where $x^2 + bx + c$ cannot be factorised	Ax-a+Bx+cx2+bx+c

Here, *A*, *B*, *C* are constants that are to be determined.

Example: Integrate the function $\frac{8x^2-17x+11}{(2x-3)(2x^2-4x+5)}$.

Solution:

Let $\frac{8x^2-17x+11}{(2x-3)(2x^2-4x+5)} = \frac{A}{2x-3} + \frac{Bx+C}{2x^2-4x+5}$ ...(1)

where A, B and C are constants.

$$\frac{8x^2-17x+11}{(2x-3)(2x^2-4x+5)} = \frac{A(2x^2-4x+5)+(Bx+C)(2x-3)}{(2x-3)(2x^2-4x+5)}$$

$$= \frac{2(A+B)x^2+(-4A-3B+2C)x+(5A-3C)}{(2x-3)(2x^2-4x+5)}$$

Comparing L.H.S. and R.H.S. of the above equation, we obtain

$2(A+B) = 8 \Rightarrow A+B = 4$ … (2)

$-4A - 3B + 2C = -17$ … (3)

and, $5A - 3C = 11$ … (4)

Solving equations (2), (3), and (4), we obtain

$A = 1$, $B = 3$, and $C = -2$

Substituting these values in equation (1), we obtain

∫12x-3+3x-22x2-4x+5dx=∫12x-3+3x-3+12x2-4x+5dx=∫12x-3+3x-32x2-4x+5+∫12x2-4x+5dx

I = I1+I2+I3

I1=∫12x-3dxI2=∫3x-32x2-4x+5dxI3=∫12x2-4x+5dx

On solving I_1, we get I1=∫12x-3dx = 12log2x-3+c1

On solving I_2, we get

I2=∫3x-32x2-4x+5dxlet 2x2-4x+5 = t4(x-1)dx=dttherefore I2=34logt+c2 = 34log2x2-4x+5+c2

On solving I_3,

I3=∫12x2-4x+5dx=∫1x2-2x+52dx = ∫1(x-1)2 + 322dx=23tan-123(x-1)+c3

So, ∫8x2-17+11(2x-3)(2x2-4x+5)dx = 12log2x-3 +34log2x2-4x+5+23tan-123(x-1)+C